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On statistical aspects of deterministic tree-like fractals

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Abstract. We prove convergence of the partition function of skewed mass distribution for a class of 'finite type' tree-like fractals.

1. Introduction

In a previous paper [10], a detailed study of a deterministic branching process was carried out. The point of view developed there was multifractal analysis of singular measures generated by asynchronous splitting on a tree. In that model, every individual in the cascade had its own 'internal splitting time', and no 'sterility' was allowed. Thermodynamic formalism applied well to this special case, in the sense that renormalization of the partition function was valid for such trees. The ideas used there stem from the theory of deterministic branching processes [8, 9, 16].

This situation is characteristic of self-similar Cantor sets in the sense of [15], where 'finite-type' (or periodic in the sense of [11]) tree structures associated with irreducible transfer matrices are considered in detail (see also [6]). Care is vital when more general structures [1, 11] are considered.

The problem we want to discuss here is the following: assume that an initial individual of unit mass is split into M sub-individuals, each of them receiving from its ancestor the sub-mass π_j ($j = 1, \ldots, M$). Assume that two kinds of offspring are now available, namely

(i) Those which will repeat the multiplicative programme of their ancestor at some (integer) instants.
(ii) Those (sterile) which will not breed.

For those populations, we ask the following questions:

(i) What is the mass distribution of such populations, for large n?
(ii) In what sense does the thermodynamic formalism apply to this new situation, in particular does a thermodynamic limit exist?
(iii) What are the natural heterogeneity characteristics of the mass distribution for such trees?

The purpose of this paper is to answer these questions.

The peculiarities of this last model, compared with the ones of [6, 15], basically rely upon the fact that, although one remains in the 'finite type' case, the associated transfer

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matrix is no longer irreducible, leading to results probably characteristic of this situation. Also, this model appears interesting so far as the understanding of 'tree-like fractal' is concerned, by which we mean that all scales, from macroscopic to microscopic, are present.

2. Definition of the structure

Let us define precisely the structure of interest. At time $n = 0$, some initial individual generates M offsprings. Among these M sons

- a_k ($k = 1 \ldots K^+$) will wait k time units before repeating the multiplicative program of their ancestor (K^+ integer).
- D of them (D integer) will not breed.

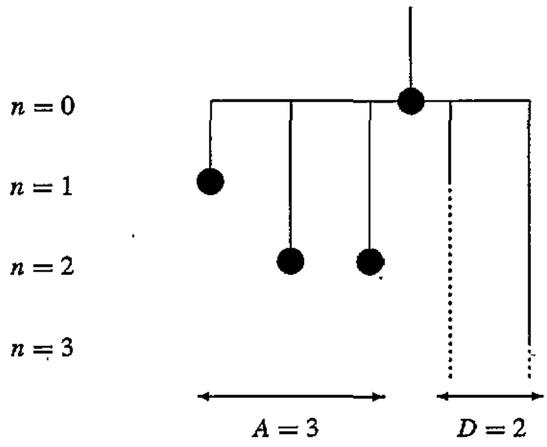
Let

$$A := \sum_{k=1}^{K^+} a_k \tag{1}$$

As a consequence $A + D = M$. We also need to introduce

$$K_- := \inf\{k : a_k \neq 0\} \tag{2}$$

This can be nicely illustrated by the following generator:



for which $a_1 = 1$, $a_2 = 2$, $a_3 = 0$, $D = 2$, $K^+ = 2$, $K_- = 1$.

Infinite branches represent the sterile individuals, while dotted ones repeat their ancestor's growth program. Note that we use the following useful convention: we order the individuals in such a way that the first to breed occur first, followed by the sterile ones.

3. The partition function of the mass distribution

3.1. Partition function

Assume that an individual of unit mass splits into M offsprings, each of them receiving the mass $\pi_j > 0$, ($j = 1 \ldots M$ and $\sum_{j=1}^{M} \pi_j = 1$). The mass splitting process is then iterated as explained in section 2.

From the point of view of statistical physics, all information on the mass distribution is contained in the partition function [6]

$$\psi_n^*(\lambda) := \sum_{i=1}^{N_n^*} \mu_i^\lambda \tag{3}$$

where N_n^* denotes the total number of individuals in this population at time n, both sterile and productive, and μ_i is the mass naturally affected to the path i in our tree, i.e.

$$\mu_i := \prod_{p=1}^{p(i)} \pi_{j_p(i)} \tag{4}$$

with $p(i) \in \{1, \ldots, p_+(n)\}$ the individual's generation number (i.e. the number of its ancestors). In this expression, $p_+(n) := n/K_-$ if n/K_- is an integer, $p_+(n) := [n/K_-] + 1$ otherwise. Also, $(\pi_{j_1(i)}, \ldots, \pi_{j_p(i)})$, $j_p(i) \in \{1, \ldots, M\}$, $p \in \{1, \ldots, p(i)\}$ is the natural 'code' for i.

Letting $N_n^*(p)$ be the amount of individuals among the N_n^* with generation number p, we observe that in the special case $\pi_j = 1/M$, $j = 1, \ldots, M$, then $\psi_n^*(\lambda)$ reads, for real λ

$$\psi_n^*(\lambda) = \sum_{p=1}^{p_+(n)} \sum_{i=1}^{N_n^*(p)} M^{-\lambda p} = \sum_{p=1}^{p_+(n)} M^{-\lambda p} N_n^*(p) \tag{5}$$

i.e. is the generating function for $N_n^*(p)$. Therefore, for any individual in this particular tree the knowledge of its generation number, say p, suffices to determine its mass, namely M^{-p}.

For the more general mass splitting process considered in this section, in order to determine the mass of a particular cylinder, one also needs to know the 'history' of the embranching.

We now come to the problem of computing $\psi_n^*(\lambda)$. For $k = 1, \ldots, K^+$, take

$$a_k(\lambda) := \sum_{j=1}^{A} \mathbb{1}_{[d(j)=k]} \pi_j^\lambda \tag{6}$$

$$D_\lambda := \left(\sum_{j=A+1}^{M} \pi_j^\lambda, 0 \ldots 0 \right) \qquad \text{a } K^+ \text{ vector} \tag{7}$$

(where the symbol $d(j)$ denotes the depth of branch $j \in \{1, \ldots, A\}$ of the generator).

Define the $K^+ \times K^+$ matrix

$$A_\lambda := \begin{bmatrix} \vdots & 1 & \cdots & 0 \\ a_k(\lambda) & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ \vdots & 0 & \cdots & 0 \end{bmatrix} \tag{8}$$

and the $(K^+ + 1) \times (K^+ + 1)$ and $(2K^+ + 2) \times (2K^+ + 2)$ matrices

$$\mathcal{A}_\lambda := \begin{bmatrix} A_\lambda & 0 \\ D_\lambda & 0 \end{bmatrix} \qquad \aleph_\lambda := \begin{bmatrix} \mathcal{A}_\lambda & 0 \\ \mathcal{A}_\lambda & I \end{bmatrix}. \tag{9}$$

Let $\mathcal{R}^t$ denote the $2K^+ + 2$ vector

$$\mathcal{R}^t := (\underbrace{1 \ldots 1}_{K^+}; \underbrace{0}_{1} \,|\, \underbrace{0 \ldots 0}_{K^+}; \underbrace{1}_{1}) \tag{10}$$

and X_0^t the $K^+ + 1$ vector

$$X_0^t := (\underbrace{1; 0 \ldots 0}_{K^+}; 0) . \tag{11}$$

Using this notation, we have the following results.

Theorem 1. $\psi_n^*(\lambda)$ as given by (3) satisfies $\psi_n^*(0) = N_n^*$, the number conservation equation, $\psi_n^*(1) = 1$, the mass conservation equation, and

$$\begin{aligned} Y_0^t(\lambda) &= (X_0^t, X_0^t) \\ Y_{n+1}(\lambda) &:= \aleph_\lambda Y_n(\lambda) \\ \psi_n^*(\lambda) &= \mathcal{R}^t Y_n(\lambda) . \end{aligned} \tag{12}$$

Proof. In order to take into account the sterile individuals, one needs to work on the cumulative process

$$\overline{X}_n(\lambda) := \sum_{m \leqslant n} X_m(\lambda) \tag{13}$$

where $X_n(\lambda)$ is the $K^+ + 1$ vector solution to

$$X_{n+1}(\lambda) = \mathcal{A}_\lambda X_n(\lambda) . \tag{14}$$

Observing that

$$\overline{X}_{n+1}(\lambda) = \overline{X}_n(\lambda) + \mathcal{A}_\lambda X_n(\lambda) \tag{15}$$

and defining the $2K^+ + 2$ column vector $Y_n(\lambda)$ by

$$Y_n^t(\lambda) := (X_n^t(\lambda), \overline{X}_n^t(\lambda)) \tag{16}$$

one obtains the recurrence for $Y_n(\lambda)$ in theorem 1. Picking up those sterile elements that remain behind, and adding up those living at the current instant n allows to build the quantity $\psi_n^*(\lambda)$ of interest. This justifies the choice of $\mathcal{R}$. □

Corollary 1. $\psi_n^*(\lambda)$ as given by (3) is the solution to the recurrent equation, for real λ

$$\begin{aligned} \psi_{n+1}^*(\lambda) &= \psi_n^*(\lambda) + \left(\sum_{j=1}^{M} \pi_j^\lambda - 1 \right) (A_\lambda^n)_{1,1} \\ \psi_0^*(\lambda) &= 1 \end{aligned} \tag{17}$$

Proof. This comes directly from theorem 1 and the immediate identity

$$\mathcal{R}^t \aleph_\lambda = \mathcal{R}^t + \left(\sum_{j=1}^{M} \pi_j^\lambda - 1 \right) \mathcal{R}_0^t \tag{18}$$

with $\mathcal{R}_0^t := (1; 0 \ldots ; 0)$. □

3.2. Asymptotics

We now want to derive the asymptotic behaviour of $\psi_n^*(\lambda)$ for large n. As is shown in corollary 1, one needs first to compute the (1, 1) term in A_λ^n as given by (8). Let $z_n(\lambda) := (A_\lambda^n)_{1,1}$. We have the following lemma.

Lemma 1. $z_n(\lambda)$ is the solution of

$$\begin{aligned} Z_0(\lambda) &= (1, 0, \ldots, 0)^t \\ Z_{n+1}(\lambda) &= A_\lambda Z_n(\lambda) \\ z_n(\lambda) &= (1, 0, \ldots, 0) Z_n(\lambda) . \end{aligned} \tag{19}$$

Let $\alpha(\lambda)$ (defined for real λ) be the positive real, strictly decreasing and strictly dominant eigenvalue of the primitive matrix A_λ, satisfying $\alpha(0) = \alpha$ and $\lim_{\lambda\to+\infty} \alpha(\lambda) = 0$. Then the s-transform $\hat{z}_\lambda(s)$of the series $((z_n(\lambda))_{n\geqslant 0}$ is well-defined, and provided that $s > \alpha(\lambda)$

$$\hat{z}_\lambda(s) := \sum_{n\geqslant 0} z_n(\lambda) s^{-n} = \left(1 - \sum_{k=1}^{K^+} s^{-k} \sum_{j=1}^{A} \mathbb{1}_{[d(j)=k]} \pi_j^\lambda\right)^{-1} \tag{20}$$

Proof. The alternative representation to (19) is the immediate renewal equation (for n integer)

$$z_n(\lambda) = \sum_{k=1}^{K^+} \left(0 \times \mathbb{1}_{[k\geqslant n]} + \left(\sum_{j=1}^{A} \mathbb{1}_{[d(j)=k]} \pi_j^\lambda\right) z_{n-k}(\lambda)\, \mathbb{1}_{[k<n]}\right) \qquad z_0(\lambda) = 1 . \tag{21}$$

Premultiplying this equation by s^{-n} and summing up over n gives the announced result. The announced proterties of $\alpha(\lambda)$ have been derived in [15]. The convergence domain of (20) is straightforward. □

We now want to compute, for $s > \alpha(\lambda)$

$$\tilde{\phi}_\lambda(s) := \sum_{n\geqslant 0} \psi_n^*(\lambda) s^{-n} . \tag{22}$$

We have the foolowing lemma.

Lemma 2. Provided that $s > \alpha(\lambda)$

$$\tilde{\phi}_\lambda(s) = \frac{1}{s-1}\left(s - \left(1 - \sum_{j=1}^{M} \pi_j^\lambda\right)\left(1 - \sum_{k=1}^{K^+} s^{-k} \sum_{j=1}^{A} \mathbb{1}_{[d(j)=k]} \pi_j^\lambda\right)^{-1}\right) \tag{23}$$

Proof. This comes directly from corollary 1 and lemma 1. □

Theorem 2. The limit $\lim_{n\to\infty} \psi_n^*(\lambda)$ of the partition function (3) exists provided $\lambda > 1$, and is

$$\psi_\infty^*(\lambda) := \left(\sum_{j=A+1}^{M} \pi_j^\lambda\right)\left(1 - \sum_{j=1}^{A} \pi_j^\lambda\right)^{-1} \tag{24}$$

This last convergence is uniform.

Proof. From lemmas 1 and 2 and from the Abelian theorem

$$\psi_{\infty}^{*}(\lambda) = \lim_{s\to 1^{+}} (s-1)\tilde{\phi}_{\lambda}(s) . \tag{25}$$

From corollary 1

$$\psi_{n}^{*}(\lambda) - \psi_{n+1}^{*}(\lambda) = \left(1 - \sum_{j=1}^{M} \pi_j^{\lambda}\right) z_n(\lambda) := w_n(\lambda) . \tag{26}$$

As a consequence of Frobenius' theorem, we see that $w_n(\lambda)$ is positive, dominated by

$$\left(1 - \sup_{j=1\ldots M} \pi_j\right) z_n(1^{+}) \approx C\alpha(1^{+})^{n} \tag{27}$$

which is a geometrical convergent series ($\alpha(1^{+}) < 1$). Therefore $w_n(\lambda)$ is convergent, and from Weierstrass' criterion $\psi_n^{*}(\lambda)$, $\lambda > 1$, is uniformly convergent. □

Comments. The limit histogram (24) is independent of the actual values of the lifetimes of individuals in the generator, which can therefore, from this point of view, be set to one. Thus these models belong to a same class parametrized by A, D and the mass fractions which deserve estimation for practical purposes.

It is indeed of vital interest in physics to derive the actual procedure one would go through to decide whether a given distribution belongs to our model class. Although this appears to be a very complex question, let us here advance some arguments in this direction that are not intended in any way to close the subject. First, define the m-order cutset (approximation) of the theoretical histogram (24) to be

$$\psi_m^{*}(\lambda) := \sum_{j=A+1}^{M} \pi_j^{\lambda} \left(1 - \left(\sum_{j=1}^{A} \pi_j^{\lambda}\right)^{m}\right)\left(1 - \sum_{j=1}^{A} \pi_j^{\lambda}\right)^{-1}$$

$$= \sum_{j=A+1}^{M} \pi_j^{\lambda} \cdot \sum_{k=0}^{m-1} \sum_{(k_1,\ldots,k_A \geqslant 0 : \sum_{j=1}^{A} k_j = k)} \left(\prod_{j=1}^{A} \pi_j^{k_j \lambda}\right) . \tag{28}$$

Take

$$\langle A \rangle_k := \frac{(A+k-1)!}{(A-1)!} . \tag{29}$$

Then for such truncated distributions, the theoretical number of *distinct* peaks is clearly

$$P_{\mathrm{T}}(m) = D\left(\sum_{k=0}^{m-2} \frac{\langle A \rangle_k}{k!}\right) + \frac{\langle A \rangle_{m-1}}{(m-1)!}$$

$$= D\frac{(A+m-2)!}{A!(m-2)!} + \frac{(A+m-2)!}{(A-1)!(m-1)!} . \tag{30}$$

and the total numbers of peaks for $m \geqslant 2$ is

$$N_{\mathrm{T}}(m) = D\left(\sum_{k=0}^{m-2} A^{k}\right) + A^{m-1} = D\frac{A^{m-1}-1}{A-1} + A^{m-1} . \tag{31}$$

This situation should be compared with any experimental histogram, where an N_{E}-sample is considered up to the precision ε of the measurements, leading to an experimental number P_{E} of distinct peaks. As a first step, one may search the combination of parameters $A = A^{*}$, $D = D^{*}$, $m = m^{*}$ for which $(P_{\mathrm{T}}, N_{\mathrm{T}}, m)$ and $(P_{\mathrm{E}}, N_{\mathrm{E}}, \varepsilon)$ fit the best, decreasing ε, i.e.

taking finer scales into consideration to see how the peaks rearrange and appear in the various histograms. There remains to determine the best set of parameters $(\pi_j)_{j=1,\dots,(A^*+D^*)}$, μ (the unit mass) explaining the observed histogram, for example fitting the moments of both theoretical and experimental approximate histograms.

4. Renyi's entropy of the mass distribution and related problems

Recalling the mass partition function is $\psi_n^*(\lambda)$, and defining the free energy as

$$F_n^*(\lambda) := -\log_\alpha \psi_n^*(\lambda) . \tag{32}$$

Introducing the probability Gibbs' λ-measure

$$G_\lambda(i) := (\mu(i)^\lambda)\left(\sum_{i=1}^{N_n^*} \mu_i^\lambda\right)^{-1} \tag{33}$$

one obtains

$$I_n^*(\lambda) := (F_n^*)'(\lambda) = -\sum_{i=1}^{N_n^*} G_\lambda(i) \log_\alpha \mu(i) . \tag{34}$$

In particular, for $1 \leqslant i \leqslant N_n^*$

$$G_0(i) = N_n^{*-1} \qquad G_1(i) = \mu(i) \tag{35}$$

and $I_n^*(1)$ is the Shannon entropy of $\mu(i)$. The quantity $I_n^*(\lambda)$is a measure of the amount of missing information for an observer who, ignoring the 'true' distribution $G_\lambda(i)$, decides to affect his own guess $G_1(i) = \mu(i)$. The quantity is often called the imprecision function [7].

Now introducing Renyi's entropy function [6]

$$S_n^*(\lambda) := \frac{1}{\lambda - 1} F_n^*(\lambda) = -\frac{1}{\lambda - 1} \log_\alpha \psi_n^*(\lambda) \tag{36}$$

and noting that the Shannon entropy is

$$I_n^*(1^+) = S_n^*(1^+) = -\sum_{i=1}^{N_n^*} \mu(i) \log_\alpha \mu(i) \tag{37}$$

one obtains

$$\begin{aligned} (S_n^*)'(\lambda) &= \frac{1}{\lambda - 1} [I_n^*(\lambda) - S_n^*(\lambda)] \\ &= \frac{1}{\lambda - 1} [(I_n^*(\lambda) - I_n^*(1^+)) - (S_n^*(\lambda) - S_n^*(1^+))] . \end{aligned} \tag{38}$$

In this expression, one recognizes Kullback's information function

$$K_n^*(\lambda) := I_n^*(\lambda) - I_n^*(1^+) . \tag{39}$$

Theorem 2 therefore allows one to compute all these limiting quantities as measures of the anisotropy of the mass distribution in the tree under study.

In particular, it can readily be shown that the limiting behaviour of the Shannon entropy $S_n^*(1^+)$ is

$$\lim_{n\to\infty} S_n^*(1^+) = S_\infty^*(1^+) = -\left(\sum_{j=1}^{M} \pi_j \log_\alpha \pi_j\right)\left(\sum_{j=A+1}^{M} \pi_j\right)^{-1} \tag{40}$$

5. Comments and conclusions

For a deterministic branching tree with sterile individuals, the formalism of the partition function has been shown to apply, provided one keeps track of these individuals (with their mass). In this last situation, the partition function itself has been shown to possess a limiting behaviour, thereby making various asymptotic measures of the anisotropy of the mass distribution in such trees meaningful. The populations under study in this paper present the main advantage of possessing a large scale of mass distribution of their constituent elements, from macroscopic to microscopic. They should therefore serve as generic models of various natural phenomena (think of the size distribution of pebbles on a beach, galaxy clusters, porous media,...). This last property also suggests that we should call such objects 'tree-like fractals'.

These facts should be compared with the ones normally in vogue, in the overlooked field of multifractal analysis of singular measures [1, 2, 3, 5, 10, 12, 14, 15]. Intimately associated with these ideas is the multiplicative cascade, as nicely illustrated by multinomial [5] or more general measures [10]. Normally (i.e. except in the case of advanced multifractals, be they random or left-sided [13]), using box-counting arguments, the partition function itself has no limiting behaviour. Rather, some sort of 'thermodynamical limit' is known to exist, leading to a rescaled uniform partition function [2]. This fact, together with large deviations arguments [1, 4, 5], allows in the 'finite type' cases to introduce the notion of the dimension spectrum [3] for these self-similar multifractal measures, which embodies all the asymptotic statistics of Lipschitz–Hölder exponents for the limit elements.

No such things are meaningful for the multiplicative cascades under study in this paper, i.e. no renormalization of the partition function is valid; the measures are therefore not stricly self-similar and this is the meaning of the term 'tree-like fractal' introduced before.

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